Esercizio 4:

Calcolare il flusso del campo vettoriale

$$F(x,y,z) = (x, y, z^4)$$

attraverso la superficie Σ del cilindro di equazione $x^2+y^2=4$ delimitato dai piani $z=-7$ e $z=7$

SOL

Usiamo il Teorema della divergenza

$$\Rightarrow \int_\Sigma \operatorname{div}(F) = \int_\Sigma 2+4z^3 \, dx\,dy\,dz$$

Usiamo le coordinate cilindriche:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0,2] \\ y = \rho\sin(\theta) & \theta \in [0,2\pi] \\ z = z & z \in [-7,7] \end{cases}$$

$$\Rightarrow \int_0^2 \int_0^{2\pi} \int_{-7}^{7} 2\rho + 4\rho z^3 \, dz\,d\theta\,d\rho =$$

$$= \int_0^2 \int_0^{2\pi} 2\rho z + \rho z^4 \Big|_{-7}^{7} \, d\theta\,d\rho = \int_0^2 \int_0^{2\pi} 4\rho \, d\theta\,d\rho =$$

$= \int_0 \int_0 \ldots$

$$= 2\pi \int_0^2 4\rho \, d\rho = 8\pi \left(\frac{\rho^2}{2}\right)\Big|_0^2 = 16\pi$$

Facciamo una prova usando la definizione di flusso:

$$\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; -1 < z < 1 ,\; x^2 + y^2 = 4 \right\}$$

$$\Rightarrow \quad \Sigma_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z = -1 ,\; x^2 + y^2 = 4 \right\}$$

$$\Sigma_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z = 1 ,\; x^2 + y^2 = 4 \right\}$$

Parametrizzo Σ_1 e Σ_2 :

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = -1 \end{cases} \quad , \quad \varphi(x,y) = \begin{cases} x = x \\ y = y \\ z = 1 \end{cases}$$

$\psi_x = (1, 0, 0)$ $\qquad$ $\varphi_x = (1, 0, 0)$

$\psi_y = (0, 1, 0)$ $\qquad$ $\varphi_y = (0, 1, 0)$

$\psi_x \wedge \psi_y = (0, 0, 1)$ $\qquad$ $\varphi_x \wedge \varphi_y = (0, 0, 1)$

$$\Rightarrow \int F \cdot n = \int F(\psi(x,y)) \cdot N_1 + \int F(\varphi(x,y)) \cdot N_2 =$$

$$\int_{\Sigma} F\cdot n = \int_{\Sigma_1} F(\psi(x,y))\cdot N_1 + \int_{\Sigma_2} F(\psi(x,y))\cdot N_2 =$$

$$\bullet \int_{\Sigma_1} F(\psi)\cdot(-N_1) = \int_{\Sigma_1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ +1 \end{pmatrix} = \int_{\Sigma_1} 1 \, dx\, dy = \int_{\Sigma_1} dx\, dy$$

in coordinate polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0,2] \\ y = \rho\sin(\theta) & \theta \in [0,2\pi] \end{cases}$$

$$\Rightarrow \int_0^2 \int_0^{2\pi} \rho \, d\theta \, d\rho = 2\pi \left(\frac{\rho^2}{2}\right)\Big|_0^2 = 8\pi$$

$$\bullet \int_{\Sigma_2} F(\psi)\cdot N_2 = \int_{\Sigma_2} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \int_{\Sigma_2} dx\, dy$$

in polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0,2] \\ y = \rho\sin(\theta) & \theta \in [0,2\pi] \end{cases}$$

$$\Rightarrow \int_0^2 \int_0^{2\pi} \rho \, d\theta \, d\rho = 2\pi \left(\frac{\rho^2}{2}\right)\Big|_0^2 = 8\pi$$

$$\Rightarrow \int_{\Sigma} F\cdot n = 8\pi + 8\pi = 16\pi$$

$\int_{\Sigma}$

Esercizio 5:

Si consideri la superficie Σ descritta da $z = x^2 + y^2$ per $x^2 + y^2 \leq R^2$ e $x \geq 0$, $y \geq 0$

Si consideri il campo vettoriale $F(x,y,z) = (1, 0, y)$

Calcolare

(1) Il flusso di $\nabla \times F$ attraverso Σ orientato verso l'alto

(2) Calcolare la circuitazione di $\vec{F}$ lungo $\partial\Sigma^+$

SOL

1): $\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, z = x^2 + y^2,\ x^2 + y^2 \leq R^2,\ x \geq 0,\ y \geq 0 \right\}$

Parametrizzo Σ:

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = x^2 + y^2 \end{cases}$$

$$\psi_x = (1, 0, 2x)$$

$$\psi_y = (0, 1, 2y)$$

$$\psi_x \wedge \psi_y = (-2x, -2y, 1)$$

$\Psi_x \wedge \Psi_y = (-2x, -2y, 1)$

e inoltre $\nabla \times F = rot(F) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 1 & 0 & y \end{vmatrix} = (1, 0, 0)$

$$\Rightarrow \int_\Sigma \nabla \times F \cdot \hat{n} = \int_\Omega rot(F(\psi)) \cdot N_1 = \int_\Omega \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix} dx\, dy =$$

$$= \int_\Omega -2x \, dx\, dy = -2 \int_\Omega x \, dx\, dy$$

in polari: $\begin{cases} x = \rho \cos(\theta) & \rho \in [0, R] \\ y = \rho \sin(\theta) & \theta \in [0, \frac{\pi}{2}] \end{cases}$

$$\Rightarrow -2 \int_0^R \int_0^{\pi/2} \rho^2 \cos(\theta) \, d\theta \, d\rho = -2 \int_0^R \rho^2 \sin(\theta) \Big|_0^{\pi/2} = -2 \int_0^R \rho^2 =$$

$$= -2 \left(\frac{\rho^3}{3} \right) \Big|_0^R = -\frac{2}{3} R^3$$

2) : Usiamo in questo caso il Teorema di Stokes

Siccome Σ è superficie liscia con bordo $\subset A$ aperto di $\mathbb{R}^3$ e $F \in C^1(A)$, allora :

e $F \in C^1(A)$, allora:

$$\oint_{\partial\Sigma^+} F \cdot \hat{T} \, d\sigma = \int_{\Sigma} rot(F) \cdot \hat{n} \, dS = \Phi_{\Sigma}(rot(F)) = -\frac{2}{3} R^3$$

$\blacksquare$

Esercizio 6:

Calcolare il flusso del rotore di $F(x,y,z) = (xy, xy, 0)$ attraverso la regione piana $S = S_1 \cup S_2$

dove $S_1 = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, \ 0 \leq y \leq \sin(\frac{\pi}{2}x)\}$

$S_2 = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq \sqrt{2}, \ 0 \leq y \leq \sqrt{2 - x^2}\}$

Sol

$$rot(F) = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ xy & xy & 0 \end{vmatrix} = (0, 0, y - x) \stackrel{def}{=} K$$

Voglio

$$\int_S K \cdot \hat{n} = \int_{S_1} K \cdot \hat{n} + \int_{S_2} K \cdot \hat{n}$$

Parametrizzo

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = y - x \end{cases}$$

$$\psi_x = (1, 0, -1)$$

$$\psi_y = (0, 1, 1)$$

$$\psi_x \wedge \psi_y = (1, -1, 1)$$

$$\Rightarrow \int_{S_1} K \cdot \hat{n} = \int_{\Omega_1} K(\psi(x,y)) \cdot N = \int_{\Omega_1} \begin{pmatrix} 0 \\ 0 \\ y-x \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} d\gamma_1 =$$

$$\int_{S_2} K \cdot \hat{n} = \int_{\Omega_2} K(\psi(x,y)) \cdot N = \int_{\Omega_2} \begin{pmatrix} 0 \\ 0 \\ y-x \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} d\gamma_2 =$$

$$\Rightarrow_{(1)} \int_{\Omega_1} y - x = \int_0^1 \int_0^{\operatorname{sen}\left(\frac{\pi}{2}x\right)} y - x \, dy \, dx = \int_0^1 \frac{y^2}{2} - xy \Big|_0^{\operatorname{sen}\left(\frac{\pi}{2}x\right)} dx =$$

$$(2) \int_{\Omega_2} y - x = \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} y - x \, dy \, dx = \int_1^{\sqrt{2}} \frac{y^2}{2} - xy \Big|_0^{\sqrt{2-x^2}} dx =$$

Da (1): $\Rightarrow \displaystyle\int_0^1 \frac{\operatorname{sen}^2\left(\frac{\pi}{2}x\right)}{2} - x \operatorname{sen}\left(\frac{\pi}{2}x\right) dx =$

$$= \frac{1}{2}\int_0^1 \underbrace{\sin^2\left(\frac{\pi}{2}x\right)dx}_{= -\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}x\right)+\frac{x}{2}} - \int_0^1 \underbrace{x \sin\left(\frac{\pi}{2}x\right)}_{= -\frac{2}{\pi}x\cos\left(\frac{\pi}{2}x\right)+\frac{4}{\pi^2}\sin\left(\frac{\pi}{2}x\right)} dx =$$

$$= \frac{1}{2}\left(\cancel{-\frac{1}{\pi}\cos\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}x\right)} + \frac{x}{2}\right)\Bigg|_0^1 - \left(\cancel{-\frac{2}{\pi}x\cos\left(\frac{\pi}{2}x\right)} + \frac{4}{\pi^2}\sin\left(\frac{\pi}{2}x\right)\right)\Bigg|_0^1 =$$

$$= \frac{1}{2}\left(\frac{1}{2}\right) - \left(\frac{4}{\pi^2}\right) = \frac{1}{4} - \frac{4}{\pi^2}$$

da (2): $\Rightarrow \displaystyle\int_1^{\sqrt{2}} \frac{2-x^2}{2} - x\sqrt{2-x^2}\, dx = \int_1^{\sqrt{2}} 1 - \frac{x^2}{2} - x\sqrt{2-x^2}\, dx =$

$$= x - \frac{1}{2}\frac{x^3}{3} + \frac{1}{3}(2-x^2)^{3/2}\Bigg|_1^{\sqrt{2}} =$$

$$= \sqrt{2} - \frac{1}{6}(\sqrt{2})^3 - 1 + \frac{1}{6} - \frac{1}{3} = \sqrt{2} - \frac{2\sqrt{2}}{6} - 1 + \frac{1}{6} - \frac{1}{3} =$$

$$= \frac{6\sqrt{2} - 2\sqrt{2} - 6 + 1 - 2}{6} = \frac{4\sqrt{2} - 7}{6} = \frac{2\sqrt{2}}{3} - \frac{7}{6}$$

In conclusione

$$\int_C K\cdot\hat{n} = \frac{1}{4} - \frac{4}{\pi^2} + \frac{2\sqrt{2}}{3} - \frac{7}{6}$$

$$\int_S K \cdot \vec{n} = 4 \quad \pi^2 \cdot \quad 3 \quad 6$$

Esercizio 2

Si consideri il campo vettoriale $F(x,y,z) = (z^3, y^3, x^3)$
Calcolare il flusso uscente di $\vec{F}$ attraverso la sfera centrata nell'origine e raggio 4

SOL

$$\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2+y^2+z^2 = 76 \right\}$$

$$\int_{\partial\Sigma} F \cdot n = \int_{\Sigma} \mathrm{div}(F) = \int_{\Sigma} 3y^2 \, dx\,dy\,dz$$

Passiamo in coordinate sferiche:

$$\begin{cases} x = \rho \sin(\psi)\cos(\theta) & \rho \in [0,4] \\ y = \rho \sin(\psi)\sin(\theta) & \psi \in [0,\pi] \\ z = \rho\cos(\psi) & \theta \in [0,2\pi] \end{cases}$$

$$\Rightarrow \int_{\Sigma} 3y^2 \, dx\,dy\,dz = 3\int_0^4 \int_0^{\pi} \int_0^{2\pi} \rho^4 \sin^3(\psi)\sin^2(\theta) \, d\theta \, d\psi \, d\rho =$$

$$= 3\int_0^4 \int_0^{\pi} \rho^4 \sin^3(\psi) \left(\frac{1}{2}\left(\theta - \sin(\theta)\cos(\theta)\right)\right)\Big|_0^{2\pi} d\psi \, d\rho =$$

$$= 3\int_0^4 \int_0^{\pi} \pi \rho^4 \sin^3(\psi) \, d\psi \, d\rho = 3\pi \int_0^4 \rho^4 \left(\frac{1}{12}\left(\cos(3\psi) - 9\cos(\psi)\right)\right)\Big|_0^{\pi} =$$

$$= 3\pi \int_0^4 \rho^4 \left(\frac{1}{12}(-1+9-1+9)\right) d\rho =$$

$$= 3\pi \int_0^4 \frac{16}{12} \rho^4 = 4\pi \left(\frac{\rho^5}{5}\right)\Bigg|_0^4 = 4\pi \cdot \frac{4^5}{5}$$

$\boxtimes$

Esercizio 4 :

Sia $\omega = \left(\frac{2x}{x^2+y^2}\right) dx + \left(\frac{2y}{x^2+y^2}\right) dy$

(1) Dire se ω è chiusa in $\mathbb{R}^2 \setminus \{0,0\}$

(2) Calcolare $\int_{\gamma^+} \omega$ dove γ è la circonferenza centrata in $(0,2)$ e di raggio 4

SOL

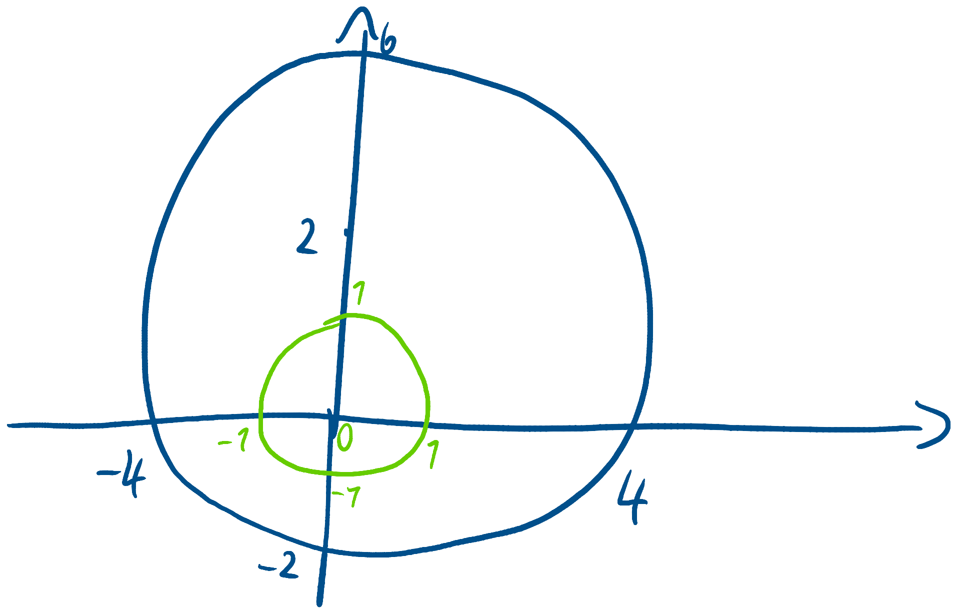
a) : $\frac{\partial}{\partial y} F_1 \overset{?}{=} \frac{\partial}{\partial x} F_2$

$$\frac{\partial}{\partial y} F_1 = -\frac{2x(2y)}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x} F_2 = -\frac{2y(2x)}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2}$$

$\Rightarrow$ è chiusa

(h):



in polari:

$$\begin{cases} x = \rho \cos(\theta) \\ y = y_0 + \rho \sin(\theta) \end{cases} \Rightarrow \begin{cases} x = 4\cos(\theta) \\ y = 2 + 4\sin(\theta) \end{cases} \quad \theta \in [0, 2\pi]$$

Consideriamo però la circonferenza centrata in (0,0) e raggio 1:

$$\begin{cases} x = \rho \cos(\theta) \\ y = \rho \sin(\theta) \end{cases} \Rightarrow \begin{cases} x = \cos(\theta) \\ y = \sin(\theta) \end{cases} \quad \theta \in [0, 2\pi]$$

$$\Rightarrow \int_{\gamma^+} \omega = \int_{\tilde{\gamma}} \omega = \int_0^{2\pi} \frac{2x}{(x^2+y^2)} \cdot \tilde{\gamma}_x' + \frac{2y}{(x^2+y^2)} \tilde{\gamma}_y' =$$

$$= \int_0^{2\pi} 2\cos(\theta)\left(-\sin(\theta)\right) + 2\sin(\theta)\cdot\cos(\theta)\, d\theta =$$

$$= \int_0^{2\pi} -2\cos(\theta)\sin(\theta) + 2\cos(\theta)\sin(\theta)\, d\theta = 0$$

$$= \int_0 -2\cos(\theta)\sin(\theta) + 2 \ldots = 0$$

$\blacksquare$

Esercizio 1 :

Dire se $M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R^3 \mid x^2+y^2+z^2-4=0 \ , \ y+x=0 \right\}$

è una varietà di R^3; determinare il punto $P \in M$ che abbia minima distanza dal punto $P_0 = (0,2,0)$

SOL

Si ha che la matrice Jacobiana è:

$$J(F) = \begin{bmatrix} 2x & 2y & 2z \\ 1 & 1 & 0 \end{bmatrix}$$

dove $F(x,y,z) = (x^2+y^2+z^2-4 \ , \ y+x)$

Calcoliamo il rango di $J(F)$

$$\det \begin{bmatrix} 2x & 2y \\ 1 & 1 \end{bmatrix} = 2x-2y = 2(x-y)$$

$$\det \begin{bmatrix} 2x & 2z \\ 1 & 0 \end{bmatrix} = -2z$$

$$\det \begin{bmatrix} 2y & 2z \\ 1 & 0 \end{bmatrix} = -2z$$

Nel primo caso ci danno fastidio i punti del tipo $y = x$

Se sostituiamo $y = x$ in M, si ha:

$$M = \{ x^2 + x^2 + z^2 - 4 = 0 \quad , \; x + x = 0 \}$$

$$\Rightarrow 2x = 0 \Rightarrow x = 0 \Rightarrow y = 0$$

$$\Rightarrow \{ z^2 = 4 \} \Rightarrow z = \pm 2$$

Quindi $z \neq 0$ e ci salva nelle altre due sottomatrici $\Rightarrow J(F)$ ha sempre rango massimo

$\Rightarrow M$ è varietà di R^3

Dobbiamo ricavare adesso il punto $P = (x, y, z)$ che abbia distanza minima da $P_0 = (0, 2, 0)$

$\Rightarrow$

Dobbiamo minimizzare $d(P, P_0) = \sqrt{(x-0)^2 + (y-2)^2 + (z-0)^2} =$

$$= \sqrt{x^2 + (y-2)^2 + z^2}$$

Dobbiamo minimizzare $f(x, y, z) = x^2 + (y-2)^2 + z^2$ sul sistema di vincoli:

$F(x, y, z) = \Rightarrow \{ x^2 + y^2 + z^2 - 4 = 0$

$$F(x,y,z)=0 \Rightarrow \begin{cases} x^2+y^2+z^2-4=0 \\ y+x=0 \end{cases}$$

Usiamo il metodo dei moltiplicatori di Lagrange

$$\begin{cases} \nabla f = \alpha \nabla F_1 + \beta \nabla F_2 \\ F_1 = 0 \\ F_2 = 0 \end{cases}$$

Ora $\nabla f = (2x,\ 2(y-2),\ 2z)$, $\nabla F_1 = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$, $\nabla F_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$

$$\begin{cases} 2x = \alpha 2x + \beta \\ 2(y-2) = \alpha 2y + \beta \\ 2z = \alpha 2z \\ x^2+y^2+z^2-4=0 \\ y+x=0 \end{cases} \Rightarrow \begin{cases} 2x = \alpha 2x + \beta \\ 2(y-2) = \alpha 2y + \beta \\ z(1-\alpha) = 0 \quad \begin{matrix} \alpha = 1 \\ z = 0 \end{matrix} \\ x^2+y^2+z^2-4=0 \\ y+x=0 \end{cases}$$

Se $\alpha = 1$:

$$\begin{cases} 2x = 2x + \beta \\ 2(y-2) = 2y + \beta \\ \alpha = 1 \\ x^2+y^2+z^2=4 \\ y+x=0 \end{cases} \Rightarrow \begin{cases} \beta = 0 \\ \beta = -4 \\ \dots \\ \dots \\ \dots \end{cases} \Rightarrow \text{IMPOSSIBILE}$$

Se $z=0$

$$\begin{cases} 2x = \alpha 2x + \beta \\ 2(y-2) = \alpha 2y + \beta \\ z=0 \\ x^2+y^2 = 4 \\ y+x = 0 \end{cases} \Rightarrow \begin{cases} 2x = \alpha 2x + \beta \\ 2(y-2) = \alpha 2y + \beta \\ z=0 \\ x^2+x^2 = 4 \\ y = -x \end{cases} \Rightarrow \begin{cases} 2x = \alpha 2x + \beta \\ 2(y-2) = \alpha 2y + \beta \\ z=0 \\ x^2 = 2 < \begin{matrix} x=+\sqrt{2} \\ x=-\sqrt{2} \end{matrix} \\ y = -x \end{cases}$$

Abbiamo due punti:

$P_1 = (-\sqrt{2}, \sqrt{2}, 0)$, $P_2 (\sqrt{2}, -\sqrt{2}, 0)$

$$f(P_1) = 2 + (\sqrt{2}-2)^2 = 8 - 4\sqrt{2}$$

$$f(P_2) = 2 + (-\sqrt{2}-2)^2 = 8 + 4\sqrt{2}$$

$f(P_1) < f(P_2)$ $\Rightarrow$ P_1 è il punto con distanza minima da $P_0 = (0,2,0)$

$\blacksquare$

Es 1:

Calcolare $\int_{\Sigma} \frac{1}{z^4}\, d\sigma$ dove $\Sigma = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3 \,\middle|\, z = \frac{1}{\sqrt{x^2+y^2}},\ 1 \le z \le 2\right\}$

SOL

Parametrizzo

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = \frac{1}{\sqrt{x^2+y^2}} \end{cases} \quad \text{e inoltre } 1 \le z \le 2 \Rightarrow 1 \ge x^2+y^2 \ge \frac{1}{4}$$

$$\psi_x = \left(1,\ 0,\ -\frac{x}{(x^2+y^2)^{3/2}}\right)$$

$$\psi_y = \left(0,\ 1,\ -\frac{y}{(x^2+y^2)^{3/2}}\right)$$

$$\psi_x \wedge \psi_y = \left(\frac{x}{(x^2+y^2)^{3/2}},\ \frac{y}{(x^2+y^2)^{3/2}},\ 1\right)$$

$$\|\ldots\| = \sqrt{\frac{x^2}{(x^2+y^2)^3} + \frac{y^2}{(x^2+y^2)^3} + 1} = \sqrt{\frac{x^2+y^2}{(x^2+y^2)^3} + 1} =$$

$$= \sqrt{\frac{1}{(x^2+y^2)^2} + 1} = \sqrt{\frac{1+(x^2+y^2)^2}{(x^2+y^2)^2}} = \frac{\sqrt{1+(x^2+y^2)^2}}{x^2+y^2}$$

$$\Rightarrow \int_{\Sigma} \frac{1}{z^4}\, d\sigma = \int_D (x^2+y^2)^2 \frac{\sqrt{1+(x^2+y^2)^2}}{x^2+y^2}\, dx\, dy =$$

$$= \int_D (x^2+y^2)\sqrt{1+(x^2+y^2)^2}\, dx\, dy$$

In coordinate polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [\frac{1}{2}, 1] \\ y = \rho\sin(\theta) & \theta \in [0, 2\pi] \end{cases}$$

$$\Rightarrow \int_{\frac{1}{2}}^{1} \int_{0}^{2\pi} \rho^3 \sqrt{1+\rho^4} \, d\theta \, d\rho =$$

$$= 2\pi \int_{\frac{1}{2}}^{1} \rho^3 \sqrt{1+\rho^4} \, d\rho = 2\pi \left[\frac{1}{6}(1+\rho^4)^{3/2} \right] \Big|_{1/2}^{1} =$$

$$= 2\pi \left(\frac{1}{6}(2)^{3/2} - \frac{1}{6}\left(1+\frac{1}{2^4}\right)^{3/2} \right) =$$

$$= \frac{2\sqrt{2}}{3}\pi - \frac{1}{6}\left(\frac{17}{16}\right)^{3/2} \cdot 2\pi = \frac{2\sqrt{2}}{3}\pi - \frac{1}{6} \cdot \frac{17}{16}\sqrt{\frac{17}{16}} \cdot 2\pi =$$

$$= \frac{2\sqrt{2}}{3}\pi - \frac{17\sqrt{17}}{192}\pi = \frac{1}{3}\pi\left(2\sqrt{2} - \frac{17}{64}\sqrt{17}\right)$$

<u>Esercizio 2</u>:

Calcolare $\int_\Sigma \frac{1}{\sqrt{1-y^4}} \, d\sigma$ dove $\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z = x + \frac{\sqrt{2}}{2} y^2 \\ 0 \le x \le \frac{\pi}{2} \\ 0 \le y \le \frac{\sqrt{2}}{2}, \; y \le \sin(x) \end{array} \right\}$

<u>SOL</u>

Parametrizzo

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = x + \frac{\sqrt{2}}{2} y^2 \end{cases}$$

$$\psi_x = (1, 0, 1)$$

$$\psi_y = (0, 1, \sqrt{2}y)$$

$$\psi_x \wedge \psi_y = (-1, -\sqrt{2}y, 1)$$

$$\Rightarrow \|\ldots\| = \sqrt{1+2y^2+1} = \sqrt{2y^2+2}$$

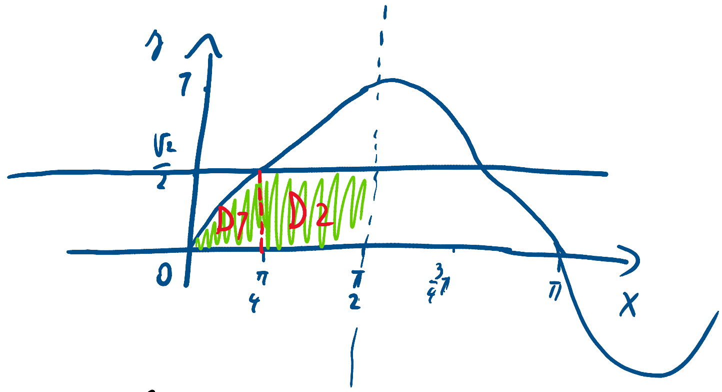
$$\Rightarrow \int_\Sigma \frac{1}{\sqrt{1-y^4}}\, d\sigma = \int_D \frac{1}{\sqrt{1-y^4}} \sqrt{2y^2+2} = \int_D \sqrt{\frac{2(y^2+1)}{(1+y^2)(1-y^2)}}\, dx\, dy =$$

$$= \int_D \sqrt{\frac{2}{1-y^2}}\, dx\, dy$$

$0 \leq x \leq \frac{\pi}{2}$

$0 \leq y \leq \frac{\sqrt{2}}{2}$

$y \leq \sin(x)$



$$\Rightarrow \int_D \sqrt{\frac{2}{1-y^2}}\, dx\, dy = \int_{D_1} \sqrt{\frac{2}{1-y^2}}\, dx\, dy + \int_{D_2} \sqrt{\frac{2}{1-y^2}}\, dx\, dy =$$

$$= \int_0^{\pi/4} \int_0^{\sin(x)} \sqrt{\frac{2}{1-y^2}}\, dy\, dx + \int_{\frac{\pi}{4}}^{\pi/2} \int_0^{\sqrt{2}/2} \sqrt{\frac{2}{1-y^2}}\, dy\, dx =$$

$$= \sqrt{2} \int_0^{\pi/4} \int_0^{\sin(x)} \frac{1}{\sqrt{1-y^2}}\, dy\, dx + \sqrt{2} \int_{\frac{\pi}{4}}^{\pi/2} \int_0^{\sqrt{2}/2} \frac{1}{\sqrt{1-y^2}}\, dy\, dx =$$

$$= \sqrt{2} \int_0^{\pi/4} \left(\arcsin(y)\right)\Big|_0^{\sin(x)} dx + \sqrt{2} \int_{\pi/4}^{\pi/2} \left[\arcsin(y)\right]\Big|_0^{\sqrt{2}/2} dx =$$

$$= \sqrt{2} \int_0^{\pi/4} x\, dx + \sqrt{2} \int_{\pi/4}^{\pi/2} \frac{\pi}{4}\, dx$$

$$= \sqrt{2} \left(\frac{x^2}{2}\right)\Big|_0^{\pi/4} + \frac{\sqrt{2}}{4}\pi\, (x)\Big|_{\pi/4}^{\pi/2} =$$

$$= \sqrt{2}\left(\frac{\pi^2}{32}\right) + \frac{\sqrt{2}}{4}\pi\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{32}\pi^2 + \frac{\sqrt{2}}{16}\pi^2 = \frac{3\sqrt{2}}{32}\pi^2$$

$\blacksquare$

Esercizio 3:

Calcolare $\int_\Sigma z^2 \, d\sigma$ dove $\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} z = xy \\ 0 \le y \le \sqrt{3}x \\ x^2+y^2 \le 1 \end{array} \right\}$

Sol

Parametrizzo

$$\varphi(x,y) = \begin{cases} x = x \\ y = y \\ z = xy \end{cases}$$

$\varphi_x = (1, 0, y)$

$\varphi_y = (0, 1, x)$

$\varphi_x \wedge \varphi_y = (-y, -x, 1)$

$\|\ldots\| = \sqrt{1+x^2+y^2}$

$\Rightarrow$

$$\int_\Sigma z^2 \, d\sigma = \int_D x^2 y^2 \sqrt{1+x^2+y^2} \, dx \, dy$$

in polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0,1] \\ y = \rho\sin(\theta) & \theta \in [???] \end{cases}$$

$0 \le y \le \sqrt{3}x \Rightarrow \rho\sin(\theta) \le \sqrt{3}\rho\cos(\theta) \Rightarrow 0 \le \sin(\theta) \le \sqrt{3}\cos(\theta)$

$\Rightarrow \operatorname{tg}(\theta) \le \sqrt{3} \Rightarrow \theta \le \operatorname{tg}^{-1}(\sqrt{3}) \Rightarrow \theta \le \frac{\pi}{3}$

$\Rightarrow 0 \le \theta \le \frac{\pi}{3}$

$$\Rightarrow \int_D x^2y^2\sqrt{1+x^2+y^2} = \int_0^1 \int_0^{\pi/3} \rho^5 \sin^2(\theta)\cos^2(\theta)\sqrt{1+\rho^2} \, d\theta \, d\rho$$

$$= \int_0^1 \rho^5 \sqrt{1+\rho^2}\, d\rho \cdot \int_0^{\pi/3} \sin^2(\theta) \underbrace{\cos^2(\theta)}_{= 1-\sin^2(\theta)}\, d\theta$$

$$= \int_0^1 \underbrace{\rho^4}_{f(x)} \underbrace{\rho\sqrt{1+\rho^2}}_{g'(x)}\, d\rho \cdot \int_0^{\pi/3} \underbrace{\sin^2(\theta)}_{\frac{1}{2}(x-\sin(x)\cos(x))} - \underbrace{\sin^4(\theta)}_{\frac{1}{32}(12x-8\sin(2x)+\sin(4x))}\, d\theta$$

$$g(x) = \frac{(1+\rho^2)^{3/2}}{3}$$

$$= \rho^4 \frac{(1+\rho^2)^{3/2}}{3} - \int 4\rho^3 \frac{(1+\rho^2)^{3/2}}{3} = \rho^4\frac{(1+\rho^2)^{3/2}}{3} - \frac{4}{3}\int \underbrace{\rho^2}_{f(x)} \underbrace{\rho(1+\rho^2)^{3/2}}_{g'(x)}$$

$$g(x) = \frac{(1+\rho^2)^{5/2}}{5}$$

$$= \rho^4 \frac{(1+\rho^2)^{3/2}}{3} - \frac{4}{3}\left(\rho^2\frac{(1+\rho^2)^{5/2}}{5} - \int 2\rho\, \frac{(1+\rho^2)^{5/2}}{5}\right) =$$

$$= \rho^4 \frac{(1+\rho^2)^{3/2}}{3} - \frac{4}{15}\rho^2(1+\rho^2)^{5/2} + \frac{4}{15}\int 2\rho(1+\rho^2)^{5/2} =$$

$$= \rho^4\frac{(1+\rho^2)^{3/2}}{3} - \frac{4}{15}\rho^2(1+\rho^2)^{5/2} + \frac{4}{15}\,\frac{2}{7}(1+\rho^2)^{7/2}$$

$$= \left(\rho^4\frac{(1+\rho^2)^{3/2}}{3} - \frac{4}{15}\rho^2(1+\rho^2)^{5/2} + \frac{8}{105}(1+\rho^2)^{7/2}\right)\Bigg|_0^1 \cdot \left[\frac{1}{2}(x-\sin(x)\cos(x)) - \frac{1}{32}(12x - 8\sin(2x)+\sin(4x))\right]_0^{\pi/3} =$$

$$\left(\frac{2\sqrt{2}}{3} - \frac{16\sqrt{2}}{15} + \frac{64\sqrt{2}}{105} - \frac{8}{105}\right)\left(\frac{1}{2}\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)\right) - \frac{1}{32}\left(12\cdot\frac{\pi}{3} - 8\sin\left(\frac{2}{3}\pi\right) + \sin\left(\frac{4\pi}{3}\right)\right)\right)$$

$$= \dots = \frac{1}{420}(11\sqrt{2}-4)\left(\frac{\pi}{3}+\frac{\sqrt{3}}{8}\right)$$

Esercizio 4:

Calcolare $\int_\Sigma x^2+y^2\, d\sigma$ dove $\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, z = \sqrt{x^2+y^2},\ x^2+y^2 < 1 \right\}$

SOL

Parametrizzo

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = \sqrt{x^2+y^2} \end{cases}$$

$$\psi_x = \left(1,\ 0,\ \frac{x}{\sqrt{x^2+y^2}}\right)$$

$$\psi_y = \left(0,\ 1,\ \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\psi_x \wedge \psi_y = \left(-\frac{x}{\sqrt{x^2+y^2}},\ -\frac{y}{\sqrt{x^2+y^2}},\ 1\right)$$

$$\|\ldots\| = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{2}$$

$$\Rightarrow \int_D (x^2+y^2)\sqrt{2} = \sqrt{2}\int_D x^2+y^2 \, dx\, dy$$

in coordinate polari:

$$\begin{cases} x = \rho \cos(\theta) & \rho \in [0,1] \\ y = \rho \sin(\theta) & \theta \in [0,2\pi] \end{cases}$$

$$\Rightarrow \sqrt{2}\int_0^1 \int_0^{2\pi} \rho^3 \, d\theta\, d\rho = \sqrt{2}\int_0^1 \rho^3(2\pi - 0)\, d\rho =$$

$$= 2\sqrt{2}\pi \int_0^1 \rho^3 \, d\rho = 2\sqrt{2}\pi \left(\frac{\rho^4}{4}\right)\Bigg|_0^1 =$$

$$= \frac{2\sqrt{2}}{4}\pi = \frac{\sqrt{2}}{2}\pi$$

Esercizio 5:

Calcolare $\int_c \frac{x^2+y^2}{z^3}$

Σ

dove $\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; (x,y,z) = \left(\sin(uv), \cos(uv), u\right) \text{ con } (u,v) \in \Omega \right\}$

e $\Omega = \left\{ (u,v) \in \mathbb{R}^2 \;\middle|\; \frac{1}{2} < u < v \;,\; v < 1 \right\}$

SOL

Parametrizzo

$$\psi(x,y) = \begin{cases} x = \sin(xy) \\ y = \cos(xy) \\ z = x \end{cases}$$

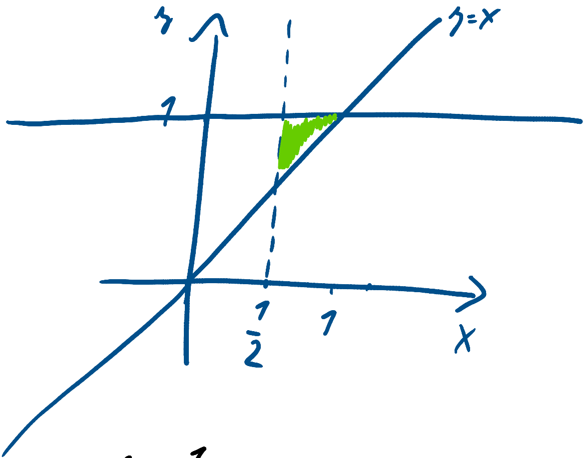
$$\psi_x = \left(y \cos(xy), -y \sin(xy), 1\right)$$

$$\psi_y = \left(x \cos(xy), -x \sin(xy), 0\right)$$

$$\psi_x \wedge \psi_y = \left(x \sin(xy), x \cos(xy), 0\right)$$

$$\|\ldots\| = \sqrt{x^2} = x$$

$$\Rightarrow \int_\Sigma \frac{x^2+y^2}{z^3}\, d\sigma = \int_\Omega \frac{1}{x^3}\, x \, dx\, dy$$



$$\frac{1}{2} < x < y$$
$$y < 1$$

$$\Rightarrow \int_\Omega \frac{1}{x^2}\, dx\, dy = \int_{1/2}^{1} \int_{x}^{1} \frac{1}{x^2}\, dy\, dx =$$

$$\int_{2} x^{2} \qquad \int_{1/2} \frac{1}{x}$$

$$= \int_{1/2}^{1} \frac{1}{x^2} \cdot y \Big|_{x}^{1} dx = \int_{1/2}^{1} \frac{1}{x^2} - \frac{1}{x} \, dx =$$

$$= \left(-\frac{1}{x} - \log(x) \right) \Big|_{1/2}^{1} = -1 + 2 - 0 + \log\left(\tfrac{1}{2}\right) =$$

$$= 1 + \log\left(\tfrac{1}{2}\right) = 1 - \log(2)$$

$\blacksquare$

Esercizio 6:

Calcolare $\displaystyle\int_{\Sigma} \frac{1}{4z+1} \, d\sigma$ dove $\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, z = x^2+y^2, \; x^2+y^2 < 1 \right\}$

SOL

Parametrizzo

$$\varphi(x,y) = \begin{cases} x = x \\ y = y \\ z = x^2+y^2 \end{cases}$$

$$\varphi_x = (1, 0, 2x)$$

$$\varphi_y = (0, 1, 2y)$$

$$\varphi_x \wedge \varphi_y = (-2x, -2y, 1)$$

$$\| \dots \| = \sqrt{1 + 4(x^2+y^2)}$$

$$\Rightarrow$$

$$\int_{\Sigma} \frac{1}{4z+1} \, d\sigma = \int_{D} \frac{1}{4(x^2+y^2)+1} \sqrt{1+4(x^2+y^2)} \, dx \, dy =$$

$$= \int_D \frac{1}{\sqrt{4(x^2+y^2)+1}} \, dx\,dy$$

in polari:

$$\begin{cases} x = \rho \cos(\theta) & \rho \in [0,1] \\ y = \rho \sin(\theta) & \theta \in [0,2\pi] \end{cases}$$

$$\int_D \frac{1}{\sqrt{4(x^2+y^2)+1}} \, dx\,dy = \int_0^1 \int_0^{2\pi} \frac{\rho}{\sqrt{4\rho^2+1}} \, d\theta\, d\rho =$$

$$= 2\pi \int_0^1 \frac{\rho}{\sqrt{4\rho^2+1}} \, d\rho = 2\pi \left(\frac{1}{4} \sqrt{4\rho^2+1} \right) \Big|_0^1 =$$

$$= \frac{\pi}{2} \left(\sqrt{5} - 1 \right)$$

$\blacksquare$

Esercizio 7 :

Calcolare l'area di

$$\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z = \frac{1}{2}(x^2+2y^2) \;,\; x^2+4y^2 < 8 \right\}$$

SOL

Parametrizzo

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = \frac{1}{2}(x^2+2y^2) \end{cases}$$

$$\psi_x = (1,\ 0,\ x)$$

$$\psi_y = (0,\ 1,\ 2y)$$

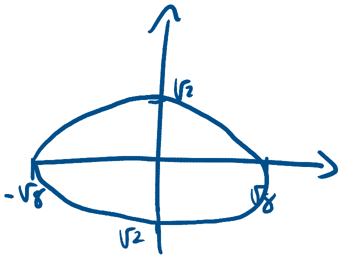
$$\psi_x \wedge \psi_y = (-x,\ -2y,\ 1)$$

$$\|\dots\| = \sqrt{1+x^2+4y^2}$$

$$\Rightarrow \int_{\Sigma} d\sigma = \int_{D} \sqrt{7+x^2+4y^2}\; dx\, dy$$

in polari:

$$\psi(\rho,\theta) = \begin{cases} x = \rho \cos(\theta) \\ y = \frac{1}{2} \rho \sin(\theta) \end{cases} \qquad \begin{array}{l} \rho \in [0, \sqrt{8}] \\ \theta \in [0, 2\pi] \end{array}$$



$$\Rightarrow \int_{D} \sqrt{7+x^2+4y^2}\; dx\, dy = \int_0^{\sqrt{8}} \int_0^{2\pi} \sqrt{7+\rho^2\cos^2(\theta)+4\left(\tfrac{1}{4}\right)\rho^2\sin^2(\theta)} \cdot \frac{\rho}{2}\, d\theta\, d\rho =$$

$$= \int_0^{\sqrt{8}} \int_0^{2\pi} \frac{\rho}{2} \sqrt{7+\rho^2}\; d\theta\, d\rho =$$

$$= \frac{1}{2} \cdot 2\pi \int_0^{\sqrt{8}} \rho \sqrt{7+\rho^2}\; d\rho = \pi \int_0^{\sqrt{8}} \rho \sqrt{7+\rho^2}\, d\rho =$$

$$= \pi \left(\frac{1}{3} (7+\rho^2)^{3/2} \right) \Bigg|_0^{\sqrt{8}} = \frac{\pi}{3} \left(7+\rho^2 \right)^{3/2} \Bigg|_0^{\sqrt{8}} =$$

$$= \frac{\pi}{3} \left((9)^{3/2} - 7 \right) = \frac{\pi}{3} (27-7) = \frac{26}{3}\pi$$

$\blacksquare$

Esercizio 8:

Calcolare l'area di

$$\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x^2+y^2+z^2 = R^2 \right\}$$

Sol

Sia $z = \pm\sqrt{R^2-x^2-y^2}$, scegliamo $z = +\sqrt{R^2-x^2-y^2}$

Parametrizzo

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = \sqrt{R^2 - x^2 - y^2} \end{cases}$$

$$\psi_x = \left(1 \, , \, 0 \, , \, -\frac{x}{\sqrt{R^2 - x^2 - y^2}}\right)$$

$$\psi_y = \left(0 \, , \, 1 \, , \, -\frac{y}{\sqrt{R^2 - x^2 - y^2}}\right)$$

$$\psi_x \wedge \psi_y = \left(\frac{x}{\sqrt{R^2 - x^2 - y^2}} \, , \, \frac{y}{\sqrt{R^2 - x^2 - y^2}} \, , \, 1\right)$$

$$\| \dots \| = \sqrt{1 + \frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2}} = \sqrt{\frac{R^2}{R^2 - x^2 - y^2}}$$

$$\int_\Sigma d\sigma = \int_D \sqrt{\frac{R^2}{R^2 - x^2 - y^2}}$$

So che $\sqrt{R^2 - x^2 - y^2} \geq 0 \Rightarrow R^2 \geq x^2 + y^2 \Rightarrow x^2 + y^2 \leq R^2$

Possiamo allora in coordinate polari:

$$\begin{cases} x = \rho \cos(\theta) & \rho \in [0, R] \\ y = \rho \sin(\theta) & \theta \in [0, 2\pi] \end{cases}$$

$$\Rightarrow \int_D R \frac{1}{\sqrt{R^2 - x^2 - y^2}} \, dx \, dy = R \int_0^R \int_0^{2\pi} \frac{\rho}{\sqrt{R^2 - \rho^2}} \, d\theta \, d\rho =$$

$$= 2\pi R \int_0^R \frac{\rho}{\sqrt{R^2 - \rho^2}} \, d\rho = 2\pi R \left(-\sqrt{R^2 - \rho^2}\right)\Big|_0^R =$$

$$= 2\pi R \, (0 + R) = 2\pi R^2$$

Questo se $z = +\sqrt{R^2 - x^2 - y^2}$

se $z = -\sqrt{R^2 - x^2 - y^2}$, il risultato è analogo se $z = -\sqrt{R^2-x^2-y^2}$, il risultato è analogo

Quindi basta moltiplicare per due il risultato ottenuto:

$$\Rightarrow \int_{\Sigma} d\sigma = 2\left(2\pi R^2\right) = 4\pi R^2$$

$\blacksquare$

Es 1:

Calcolare il flusso uscente del campo vettoriale $F(x,y,z) = (x, y, z^2)$ dalla superficie costituita dal bordo di $D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid -1 < z < -x^2 - y^2 \right\}$

Sol

Il flusso può essere calcolato in 2 modi:

1) Con la definizione
2) Col teorema della divergenza

Vediamo entrambi i metodi:

1° Metodo: Con la definizione

Si ha che $\partial D = \Sigma_1 \cup \Sigma_2$

dove $\Sigma_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid z = -1, \quad x^2 + y^2 < 1 \right\}$

$\Sigma_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid z = -x^2 - y^2, \quad x^2 + y^2 < 1 \right\}$

Quindi:

$$\int_{\partial D} F \cdot n = \int_{\Sigma_1} F \cdot n + \int_{\Sigma_2} F \cdot n$$

Parametrizzo Σ_1 e Σ_2 :

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = -1 \end{cases} \quad , \quad \varphi(x,y) = \begin{cases} x = x \\ y = y \\ z = -x^2 - y^2 \end{cases}$$

$\psi_x = (1, 0, 0)$ $\qquad\qquad$ $\varphi_x = (1, 0, -2x)$

$\psi_y = (0, 1, 0)$ $\qquad\qquad$ $\varphi_y = (0, 1, -2y)$

$\psi_x \wedge \psi_y = (0,0,1)$ $\qquad\qquad$ $\varphi_x \wedge \varphi_y = (2x, 2y, 1)$

Per definizione di integrale di flusso si ha che:

$$\int_{\Sigma_1} F\cdot n = \int_D F(\psi(x,y)) \cdot N_1(x,y)\, dx\, dy$$

dove $N_1 = \psi_x \wedge \psi_y = (0,0,1)$

Però questo N_1 è entrante in D

$\Rightarrow -N_1 = (0,0,-1)$ è uscente

Ne segue che:

$$\int_{\Sigma_1} F\cdot n = \int_D F(\psi(x,y)) \cdot \begin{pmatrix}0\\0\\-1\end{pmatrix} dx\, dy = \int_D \begin{pmatrix}x\\y\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\-1\end{pmatrix} dx\, dy =$$

$$= \int_D -1\, dx\, dy = -\int_D dx\, dy$$

in polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0,1] \\ y = \rho\sin(\theta) & \theta \in [0, 2\pi) \end{cases}$$

$$\Rightarrow -\int_D dx\, dy = -\int_0^1\int_0^{2\pi} \rho\, d\theta\, d\rho = -2\pi \left(\frac{\rho^2}{2}\right)\Big|_0^1 = -\pi$$

Abbiamo calcolato il primo pezzo

Ora, per definizione di integrale di flusso, si ha:

$$\int_{\Sigma_2} F\cdot n = \int_D F(\varphi(x,y))\cdot N_2$$

dove $N_2 = \varphi_x \wedge \varphi_y = (2x, 2y, 1)$

In questo caso N_2 è uscente da D

Ne segue che

$$\int_D F(\varphi(x,y))\cdot N_2 = \int_D F(\varphi(x,y))\cdot\begin{pmatrix}2x\\2y\\1\end{pmatrix} dx\,dy =$$

$$= \int_D \begin{pmatrix}x\\y\\(-x^2-y^2)^2\end{pmatrix}\begin{pmatrix}2x\\2y\\1\end{pmatrix} dx\,dy = \int_D 2x^2+2y^2+(-x^2-y^2)^2\,dx\,dy =$$

in polari:

$$\begin{cases} x=\rho\cos(\theta) & \rho\in[0,1] \\ y=\rho\sin(\theta) & \theta\in[0,2\pi] \end{cases}$$

$$\Rightarrow \int_0^1\int_0^{2\pi} 2\rho^3+\rho^5\,d\theta\,d\rho = 2\pi\int_0^1 2\rho^3+\rho^5\,d\rho =$$

$$= 2\pi\left(2\frac{\rho^4}{4}+\frac{\rho^6}{6}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2}+\frac{1}{6}\right) = 2\pi\cdot\frac{4}{6} = \frac{4}{3}\pi$$

Abbiamo calcolato anche il secondo pezzo

Ora dobbiamo solamente sommarli:

$$\Rightarrow \int_{\partial D} F\cdot n = \int_{\Sigma_1} F\cdot n + \int_{\Sigma_2} F\cdot n = -\pi+\frac{4}{3}\pi = \pi\left(\frac{4}{3}-1\right) = \frac{1}{3}\pi$$

2° Metodo: Teorema della divergenza

Essendo $F_1\in C^1$ e D un aperto con bordo, per il teorema della divergenza si ha che:

$$\int_{\partial D} F\cdot n = \int_D \operatorname{div}(F)\,dx\,dy\,dz$$

Posto $F = (f_1, f_2, f_3)$ si ha che:

$$\mathrm{div}(F) = \frac{\partial f_1}{\partial x}(x,y,z) + \frac{\partial f_2}{\partial y}(x,y,z) + \frac{\partial f_3}{\partial z}(x,y,z)$$

$$= 1 + 1 + 2z = 2 + 2z = 2(1+z)$$

$$\Rightarrow \int_{\partial D} F \cdot n = \int_D \mathrm{div}(F) = \int_D 2(1+z)\, dx\, dy\, dz = 2\int_D 1+z \, dx\, dy\, dz$$

$$\Rightarrow 2\int_\Omega \int_{-1}^{-x^2-y^2} 1+z \, dz \, dx\, dy = 2\int_\Omega \left(z + \frac{z^2}{2}\right)\Bigg|_{-1}^{-x^2-y^2} dx\, dy =$$

$$= 2\int_\Omega -x^2-y^2+1+\frac{(-x^2-y^2)^2}{2} - \frac{1}{2}\, dx\, dy$$

Siccome $-1 < -x^2-y^2 \Rightarrow 1 > x^2+y^2$

in polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0,1] \\ y = \rho\sin(\theta) & \theta \in [0, 2\pi) \end{cases}$$

$$2\int_\Omega -x^2-y^2+\frac{(-x^2-y^2)^2}{2} + \frac{1}{2}\, dx\, dy =$$

$$= 2\int_\Omega \frac{2(-x^2-y^2)+(-x^2-y^2)^2+1}{2}\, dx\, dy =$$

$$= \int_\Omega -2(x^2+y^2)+(-x^2-y^2)^2+1\, dx\, dy =$$

$$= \int_0^1 \int_0^{2\pi} -2\rho^3 + \rho^5 + \rho \, d\theta \, d\rho =$$

$$= \int_0^1 \int_0^{2\pi} -2\rho^3 + \rho^5 + \rho \, d\theta \, d\rho =$$

$$= 2\pi \int_0^1 -2\rho^3 + \rho^5 + \rho = 2\pi \left(-2\frac{\rho^4}{4} + \frac{\rho^6}{6} + \frac{\rho^2}{2}\right)\Bigg|_0^1 =$$

$$= 2\pi \left(-\frac{1}{2} + \frac{1}{6} + \frac{1}{2}\right) = \frac{2}{6}\pi = \frac{1}{3}\pi$$

Es 2:

Calcolare il flusso uscente del campo F dal bordo dell'insieme D nei seguenti casi:

(a): $F(x,y,z) = (x,y,z)$ e $D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x+y+z<1 \,,\ \begin{matrix} x>0 \\ y>0 \\ z>0 \end{matrix} \right\}$

(b): $F(x,y,z) = (x^2, y^2, z)$ e $D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x^2+y^2 < z < 1 \right\}$

(c): $F(x,y,z) = (x^3, y^3, z^3)$ e $D = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x^2+y^2+z^2 < 1 \,,\ z>0 \right\}$

SOL

a): $0 < x+y+z < 1$

$0 < z < 1-x-y$

$\Sigma_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, z=0 \,,\ y<1-x \right\}$

$\Sigma_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, z=1-x-y \,,\ y<1-x \right\}$

1° Metodo:

$$\int_{\partial D} F\cdot n = \int_{\Sigma_1} F\cdot n + \int_{\Sigma_2} F\cdot n$$

Parametrizzo Σ_1 e Σ_2 :

$$\varphi(x,y) = \begin{cases} x = x \\ y = y \\ z = 0 \end{cases} \quad , \quad \psi(x,y) = \begin{cases} x = x \\ y = y \\ z = 1-x-y \end{cases}$$

$\varphi_x = (1,0,0)$ $\qquad$ $\psi_x = (1, 0, -1)$

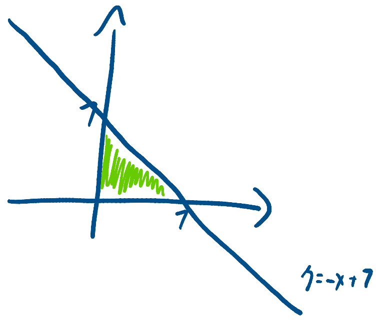
$\varphi_y = (0\ 1,0)$ $\qquad$ $\psi_y = (0, 1, -1)$

$\varphi_x \wedge \varphi_y = (0,0,1)$ $\qquad$ $\psi_x \wedge \psi_y = (1,1,1)$

$\Rightarrow$

- $$\int_{\Sigma_1} F \cdot n = \int_\Omega F(\varphi(x,y)) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\,dy = \int_\Omega \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\,dy = 0$$

- $$\int_{\Sigma_2} F \cdot n = \int_\Omega F(\psi(x,y)) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} =$$ $\quad y < 1-x \Rightarrow$



$$= \int_\Omega \begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \int_\Omega 1\,dx\,dy =$$

$$= \int_0^1 \int_0^{1-x} dy\,dx = \int_0^1 1-x\,dx = x - \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} \quad \Rightarrow \int_{\Sigma_1} + \int_{\Sigma_2} = \frac{1}{2}$$

__2° Metodo__ :

$$\int_{\partial D} F \cdot n = \int_D \mathrm{div}(F)$$

Posto $F = (f_1, f_2, f_3) \Rightarrow \mathrm{div}(F) = f_{1_x} + f_{2_y} + f_{3_z}$

$\Rightarrow \mathrm{div}(F) = 3$

$$\Rightarrow \int_{\partial D} F \cdot n = \int_D \mathrm{div}(F)\, dx\,dy\,dz = \int_D 3\, dx\,dy\,dz = 3 \int_D dx\,dy\,dz$$

$$0 < x + y + z < 1$$

$0 < x$
$0 < y$
$0 < z$
$\Rightarrow \quad 0 < z < 1 - x - y$

$0 < 1 - x - y \Rightarrow 0 < y < 1 - x$

$0 < 1 - x \Rightarrow 0 < x < 1$

$$\Rightarrow 3\int_D dx\,dy\,dz = 3\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dz =$$

$$= 3\int_0^1 \int_0^{1-x} z\Big|_0^{1-x-y} dy\,dx = 3\int_0^1 \int_0^{1-x} 1-x-y\,dy\,dx =$$

$$= 3\int_0^1 \left(y - xy - \frac{y^2}{2}\right)\Big|_0^{1-x} dx = 3\int_0^1 1-x-x(1-x) - \frac{(1-x)^2}{2}\,dx =$$

$$= 3\int_0^1 1-x-x+x^2 - \frac{1+x^2-2x}{2}\,dx = 3\int_0^1 \frac{2-4x+2x^2-1-x^2+2x}{2}\,dx =$$

$$= \frac{3}{2}\int_0^1 1-2x+x^2\,dx = \frac{3}{2}\left(x - x^2 + \frac{x^3}{3}\right)\Big|_0^1 =$$

$$= \frac{3}{2}\left(1-1+\frac{1}{3}\right) = \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2}$$

b): <u>1º Método</u>:

$$\Sigma_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z = x^2 + y^2 \;,\; x^2 + y^2 < 1 \right\}$$

$$\Sigma_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z = 1 \;,\; x^2 + y^2 < 1 \right\}$$

$$\Rightarrow \int_{\partial D} F \cdot n = \int_{\Sigma_1} F \cdot n + \int_{\Sigma_2} F \cdot n$$

Parametrizzo Σ_1 e Σ_2 :

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = x^2+y^2 \end{cases} \quad , \quad \varphi(x,y) = \begin{cases} x = x \\ y = y \\ z = 1 \end{cases}$$

$$\psi_x = (1, 0, 2x) \qquad \varphi_x = (1, 0, 0)$$

$$\psi_y = (0, 1, 2y) \qquad \varphi_y = (0, 1, 0)$$

$$\psi_x \wedge \psi_y = (-2x, -2y, 1) \qquad \varphi_x \wedge \varphi_y = (0, 0, 1)$$

Ora

dobbiamo cambiare di segno

$$\int_{\Sigma_1} F \cdot n = \int_{\Omega} F(\psi(x,y)) \cdot N_1 = \int_{\Omega} F(\psi(x,y)) \cdot \begin{pmatrix} +2x \\ +2y \\ -1 \end{pmatrix} =$$

$$= \int_{\Omega} \begin{pmatrix} x^2 \\ y^2 \\ x^2+y^2 \end{pmatrix} \begin{pmatrix} +2x \\ +2y \\ -1 \end{pmatrix} dx\,dy = \int_{\Omega} +2x^3 + 2y^3 - x^2 - y^2 \, dx\,dy$$

in polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0,1] \\ y = \rho\sin(\theta) & \theta \in [0,2\pi] \end{cases}$$

$$\int_0^1 \int_0^{2\pi} +2\rho^4 \cos^3(\theta) + 2\rho^4 \sin^3(\theta) - \rho^3 \, d\theta \, d\rho =$$

$$\int_0^1 +2\rho^4 \left(\frac{1}{12}(9\sin(\theta) + \sin(3\theta)\right)\Big|_0^{2\pi} + 2\rho^4\left(\frac{1}{12}(\cos(3\theta) - 9\cos(\theta)\right)\Big|_0^{2\pi} - \rho^3\theta\Big|_0^{2\pi} =$$

$$= -\int_0^1 2\pi\rho^3 \, d\rho = 2\pi \int_0^1 \rho^3 = -2\pi\left(\frac{\rho^4}{4}\right)\Big|_0^1 = -\frac{1}{2}\pi$$

Ora

$$\int_{\Sigma_2} F \cdot n = \int_{\Omega} F(\varphi(x,y)) \cdot N_2 = \int_{\Omega} F(\varphi(x,y)) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \int_{\Omega} \begin{pmatrix} x^2 \\ y^2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dx\,dy =$$

$$= \int_{\Omega} dx\, dy$$

in coordinate polari:

$$\begin{cases} x = \rho \cos(\theta) & \rho \in [0,1] \\ y = \rho \sin(\theta) & \theta \in [0,2\pi) \end{cases}$$

$$\Rightarrow \int_{\Omega} dx\, dy = \int_0^1 \int_0^{2\pi} \rho \, d\theta \, d\rho = 2\pi \int_0^1 \rho =$$

$$= 2\pi \left(\frac{\rho^2}{2} \right) \Big|_0^1 = \pi$$

$$\Rightarrow \int_{\Sigma_1} + \int_{\Sigma_2} = -\frac{1}{2}\pi + \pi = \frac{\pi}{2}$$

<u>2° Metodo</u>:

$$\int_{\partial D} F \cdot n = \int_D \operatorname{div}(F) \, dx\, dy\, dz$$

Posto $F = (f_1, f_2, f_3) \Rightarrow \operatorname{div}(F) = f_{1_x} + f_{2_y} + f_{3_z}$

$$\Rightarrow \operatorname{div}(F) = 2x + 2y + 1$$

$$\Rightarrow \int_D \operatorname{div}(F) = \int_D 2x + 2y + 1 \, dx\, dy\, dz$$

$$= \int_{\Omega} \int_{x^2+y^2}^{1} 2x + 2y + 1 \, dz \, dx\, dy =$$

$$= \int_{\Omega} 2xz + 2yz + z \Big|_{x^2+y^2}^{1} = \int_{\Omega} 2x + 2y + 1 - 2x(x^2+y^2) - 2y(x^2+y^2) - x^2 - y^2 \, dx\, dy$$

in polari:

$$\begin{cases} x = \rho \cos(\theta) & \rho \in [0,1] \\ y = \rho \sin(\theta) & \theta \in [0,2\pi) \end{cases}$$

$$\Rightarrow \int_0^1\int_0^{2\pi} 2\rho^2\cos(\theta)+2\rho^2\sin(\theta)+\rho-2\rho^4\cos(\theta)-2\rho^4\sin(\theta)-\rho^3 \, \partial\theta\,\partial\rho =$$

$$=\int_0^1 \cancel{2\rho^2\sin(\theta)} \cancel{-2\rho^2\cos(\theta)}+\rho\cdot\theta \cancel{-2\rho^4\sin(\theta)} \cancel{+2\rho^4\cos(\theta)}-\rho^3\theta\Big|_0^{2\pi} \partial\rho =$$

$$=\int_0^1 2\pi\rho-2\pi\rho^3 = \int_0^1 2\pi(\rho-\rho^3) = 2\pi\int_0^1 \rho-\rho^3\,\partial\rho =$$

$$=2\pi\left(\frac{\rho^2}{2}-\frac{\rho^4}{4}\right)\Big|_0^1 = 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = 2\pi\left(\frac{1}{4}\right) = \frac{\pi}{2}$$

c): <u>1° Metodo</u>

$x^2+y^2+z^2<1 \quad , \quad z>0$

$z^2<1-x^2-y^2 \Rightarrow 0<z<\sqrt{1-x^2-y^2}$

$$\Sigma_1 = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; z=0 \quad , \quad x^2+y^2<1\right\}$$

$$\Sigma_2 = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \;\middle|\; z=\sqrt{1-x^2-y^2} \quad , \quad x^2+y^2<1\right\}$$

Parametrizzo Σ_1 e Σ_2 :

$$\varphi(x,y)=\begin{cases}x=x\\y=y\\z=0\end{cases} \quad , \quad \psi(x,y)=\begin{cases}x=x\\y=y\\z=\sqrt{1-x^2-y^2}\end{cases}$$

$\varphi_x=(1,0,0) \qquad \psi_x=\left(1,0,-\frac{x}{\sqrt{1-x^2+y^2}}\right)$

$\varphi_y=(0,1,0) \qquad \psi_y=\left(0,1,-\frac{y}{\sqrt{1-x^2-y^2}}\right)$

$\varphi_x\wedge\varphi_y=(0,0,1) \qquad \psi_x\wedge\psi_y=\left(\frac{x}{\sqrt{1-x^2-y^2}},\frac{y}{\sqrt{1-x^2-y^2}},1\right)$

$$\Rightarrow \int_{\partial D} F\cdot n = \int_{\Sigma_1} F\cdot n + \int_{\Sigma_2} F\cdot n$$

$$\cdot \int_{\Sigma_1} F\cdot n = \int_{\Omega} F(\varphi(x,y))\cdot(-N_1) = \int_{\Omega} F(\varphi(x,y))\cdot\begin{pmatrix}0\\0\\-1\end{pmatrix} =$$

$$\bullet \int_{\Sigma_1} F\cdot n = \int_\Omega F(\varphi(x,y))\cdot(-N_1) = \int_\Omega F(\varphi(x,y))\cdot(-1)$$

$$= \int_\Omega \begin{pmatrix} x^3 \\ y^3 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \,dx\,dy = 0$$

$$\bullet \int_{\Sigma_2} F\cdot n = \int_\Omega F(\varphi(x,y))\cdot N_2 = \int_\Omega F(\varphi(x,y))\begin{pmatrix} \frac{x}{\sqrt{1-x^2-y^2}} \\ \frac{y}{\sqrt{1-x^2-y^2}} \\ 1 \end{pmatrix} =$$

$$= \int_\Omega \begin{pmatrix} x^3 \\ y^3 \\ \left(\sqrt{1-x^2-y^2}\right)^3 \end{pmatrix}\cdot\begin{pmatrix} \frac{x}{\sqrt{1-x^2-y^2}} \\ \frac{y}{\sqrt{1-x^2-y^2}} \\ 1 \end{pmatrix} =$$

$$= \int_\Omega \frac{x^4}{\sqrt{1-x^2-y^2}} + \frac{y^4}{\sqrt{1-x^2-y^2}} + \left(\sqrt{1-x^2-y^2}\right)^3 \,dx\,dy$$

Passiamo in coordinate polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho\in[0,1] \\ y = \rho\sin(\theta) & \theta\in[0,1] \end{cases}$$

$$\Rightarrow \int_0^1\int_0^{2\pi} \frac{\rho^5\cos^4(\theta)}{\sqrt{1-\rho^2}} + \frac{\rho^5\sin^4(\theta)}{\sqrt{1-\rho^2}} + \rho\left(\sqrt{1-\rho^2}\right)^3 d\theta$$

$$= \int_0^1\int_0^{2\pi} \frac{\rho^5\cos^4(\theta) + \rho^5\sin^4(\theta) + \rho(1-\rho^2)^2}{\sqrt{1-\rho^2}}\,d\theta\,d\rho =$$

$$= \int_0^1\int_0^{2\pi} \frac{\rho^5(\cos^4(\theta)+\sin^4(\theta)) + \rho(1+\rho^4-2\rho^2)}{\sqrt{1-\rho^2}}\,d\theta\,d\rho =$$

$$= \int_0^1\int_0^{2\pi} \frac{\rho^5(\cos^4(\theta)+\sin^4(\theta)) + \rho+\rho^5-2\rho^3}{\sqrt{1-\rho^2}}\,d\theta\,d\rho =$$

$$= \int_0^1\int_0^{2\pi} \frac{\rho^5\cos^4(\theta)}{\sqrt{1-\rho^2}} + \frac{\rho^5\sin^4(\theta)}{\sqrt{1-\rho^2}} + \frac{\rho}{\sqrt{1-\rho^2}} + \frac{\rho^5}{\sqrt{1-\rho^2}} - \frac{2\rho^3}{\sqrt{1-\rho^2}}\,d\theta\,d\rho =$$

$$\int_0 \int_0 \sqrt{1-\rho^2} \qquad \sqrt{1-\rho^2} \qquad \sqrt{1-\rho^2} \qquad \sqrt{1-\rho^2} \qquad \sqrt{1-\rho^2}$$

$$= \int_0^1 \frac{\rho^5}{\sqrt{1-\rho^2}} \left(\frac{1}{32} \left(12\theta + \cancel{8 \sin(2\theta)} + \cancel{\sin(4\theta)} \right) \right) + \frac{\rho^5}{\sqrt{1-\rho^2}} \left(\frac{1}{32} \left(12\theta - \cancel{8 \sin(2\theta)} + \cancel{\sin(4\theta)} \right) \right) + \frac{\rho}{\sqrt{1-\rho^2}}\, \theta + \frac{\rho^5}{\sqrt{1-\rho^2}}\, \theta - \frac{2\rho^3}{\sqrt{1-\rho^2}}\, \theta \, \Bigg|_0^{2\pi} =$$

$$= \int_0^1 \frac{\rho^5}{\sqrt{1-\rho^2}} \left(\frac{1}{32} \cdot 12 \cdot 2\pi \right) + \frac{\rho^5}{\sqrt{1-\rho^2}} \left(\frac{1}{32} \cdot 12 \cdot 2\pi \right) + \frac{\rho}{\sqrt{1-\rho^2}}\, 2\pi + \frac{\rho^5}{\sqrt{1-\rho^2}}\, 2\pi - \frac{\rho^3}{\sqrt{1-\rho^2}}\, 2\pi \, d\rho =$$

$$= \int_0^1 \frac{48}{32}\pi \frac{\rho^5}{\sqrt{1-\rho^2}} + 2\pi \frac{\rho}{\sqrt{1-\rho^2}} + 2\pi \frac{\rho^5}{\sqrt{1-\rho^2}} - 2\pi \frac{\rho^3}{\sqrt{1-\rho^2}} \, d\rho$$

$$= \int_0^1 \frac{112}{32}\pi \underbrace{\frac{\rho^5}{\sqrt{1-\rho^2}}}_{= -\rho^4\sqrt{1-\rho^2} - \frac{4}{3}\rho^2(1-\rho^2)^{3/2} - \frac{2}{15}(1-\rho^2)^{5/2}} - 2\pi \overbrace{\frac{\rho^3}{\sqrt{1-\rho^2}}}^{-\rho^2\sqrt{1-\rho^2} - \frac{2}{3}(1-\rho^2)^{3/2}} + 2\pi \overbrace{\frac{\rho}{\sqrt{1-\rho^2}}}^{-\sqrt{1-\rho^2}} \, d\rho \quad =$$

$$= \frac{112}{32}\pi \left(\cancel{-\rho^4\sqrt{1-\rho^2}} - \cancel{\frac{4}{3}\rho^2(1-\rho^2)^{3/2}} - \frac{2}{15}(1-\rho^2)^{5/2} \right) \Bigg|_0^1 \quad +$$

$$-2\pi \left(\cancel{-\rho^2\sqrt{1-\rho^2}} - \frac{2}{3}(1-\rho^2)^{3/2} \right) \Bigg|_0^1 \; +$$

$$+ 2\pi \left(-\sqrt{1-\rho^2} \right) \Big|_0^1 \qquad =$$

$$= \frac{112}{32}\pi \left(\frac{2}{15} \right) - 2\pi \left(\frac{2}{3} \right) + 2\pi \left(1 \right) = \frac{112}{32} \cdot \frac{2}{15}\pi - \frac{4}{3}\pi + 2\pi =$$

$$= \frac{17}{15}\pi$$

2° Metodo :

$$\int_{\partial D} F \cdot n = \int_D \operatorname{div}(F) \, dx \, dy \, dz$$

Posto $F = (f_1, f_2, f_3) \Rightarrow \operatorname{div}(F) = f_{1x} + f_{2y} + f_{3z} \Rightarrow$

$\Rightarrow \operatorname{div}(F) = 3x^2 + 3y^2 + 3z^2$

$$\Rightarrow \int_D 3x^2+3y^2+3z^2$$

$0<z<\sqrt{1-x^2-y^2}$

$\Rightarrow 1-x^2-y^2>0 \Rightarrow x^2+y^2<1$

$$\Rightarrow \int_\Omega \int_0^{\sqrt{1-x^2-y^2}} 3x^2+3y^2+3z^2 \, dz \, dx \, dy =$$

$$= \int_\Omega 3x^2 z + 3y^2 z + z^3 \Big|_0^{\sqrt{1-x^2-y^2}} dx\, dy =$$

$$= \int_\Omega 3x^2\sqrt{1-x^2-y^2} + 3y^2\sqrt{1-x^2-y^2} + (1-x^2-y^2)\sqrt{1-x^2-y^2} \; dx\, dy$$

in polari: $\begin{cases} x=\rho\cos(\theta) & \rho\in[0,1] \\ y=\rho\sin(\theta) & \theta\in[0,2\pi) \end{cases}$

$$\Rightarrow \int_0^1\int_0^{2\pi} 3\rho^3\cos^2(\theta)\sqrt{1-\rho^2} + 3\rho^3\sin^2(\theta)\sqrt{1-\rho^2} + \rho(1-\rho^2)\sqrt{1-\rho^2} \; d\theta\, d\rho$$

$$= \int_0^1\int_0^{2\pi} 3\rho^3\sqrt{1-\rho^2} + (\rho-\rho^3)\sqrt{1-\rho^2} \; d\theta\, d\rho =$$

$$= 2\pi\int_0^1 3\rho^3\sqrt{1-\rho^2} + \rho\sqrt{1-\rho^2} - \rho^3\sqrt{1-\rho^2} \; d\rho$$

$$= 2\pi\int_0^1 2\rho^3\sqrt{1-\rho^2} + \underbrace{\rho\sqrt{1-\rho^2}}_{-\frac{1}{3}(1-\rho^2)^{3/2}} \; d\rho =$$

$$= 2\pi\left(-\frac{1}{3}\cancel{\rho^2(1-\rho^2)}^{3/2} - \frac{2}{15}(1-\rho^2)^{5/2} - \frac{1}{3}(1-\rho^2)^{3/2}\right)\Big|_0^1 =$$

$$= 2\pi\left(+\frac{2}{15}+\frac{1}{3}\right) = \frac{14}{15}\pi$$

2° Metodo (ma con coordinate sferiche):

$$\int_{\partial D} F \cdot n = \int_D \operatorname{div}(F) = \int_D 3x^2 + 3y^2 + 3z^2$$

in sferiche:

$$\begin{cases} x = \rho \sin(\psi)\cos(\theta) & \rho \in [0,1] \\ y = \rho \sin(\psi)\sin(\theta) & \psi \in [0,\pi/2] \\ z = \rho\cos(\psi) & \theta \in [0,2\pi] \end{cases}$$

$$\Rightarrow \int_0^1 \int_0^{2\pi} \int_0^{\pi/2} 3\rho^4 \sin(\psi)\, d\psi\, d\theta\, d\rho =$$

$$= \int_0^1 \int_0^{2\pi} -3\rho^4 \cos(\psi)\Big|_0^{\pi/2} d\theta\, d\rho =$$

$$= \int_0^1 \int_0^{2\pi} 3\rho^4\, d\theta\, d\rho = 3 \cdot 2\pi \int_0^1 \rho^4\, d\rho =$$

$$= 6\pi \left(\frac{\rho^5}{5}\right)\Big|_0^1 = \frac{6}{5}\pi$$

■

Esercizio:

Calcolare il flusso di $F(x,y,z) = (-3x, -3y, 2\sqrt{x^2+y^2} - z)$ attraverso la superficie $\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, z = 2\sqrt{x^2+y^2},\ x^2+y^2 \leq 9 \right\}$ orientata in modo che il versore normale a Σ formi un angolo ottuso con il versore fondamentale dell'asse z.

SOL

Parametrizzo Σ:

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = 2\sqrt{x^2+y^2} \end{cases}$$

$$\psi_x = \left(1, 0, \frac{2x}{\sqrt{x^2+y^2}}\right)$$

$$\psi_y = \left(0, 1, \frac{2y}{\sqrt{x^2+y^2}}\right)$$

$$\psi_x \wedge \psi_y = \left(-\frac{2x}{\sqrt{x^2+y^2}}, -\frac{2y}{\sqrt{x^2-y^2}}, 1\right)$$

Ora notiamo che

$$\left\langle N, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = 1 > 0 \implies \text{è acuto}$$

Prendiamo allora il vettore $-N \implies \left(\frac{2x}{\sqrt{x^2+y^2}}, \frac{2y}{\sqrt{x^2+y^2}}, -1\right)$

$$\implies \int_{\partial\Sigma} F \cdot n = \int_{\Sigma} F(\psi) \cdot N = \int_{\Sigma} \begin{pmatrix} -3x \\ -3y \\ 0 \end{pmatrix} \left(\frac{2x}{\sqrt{x^2+y^2}}, \frac{2y}{\sqrt{x^2+y^2}}, -1\right) =$$

$$= \int_{\Omega} -\frac{6x^2}{\sqrt{x^2+y^2}} - \frac{6y^2}{\sqrt{x^2+y^2}} \, dx\, dy = \int_{\Omega} -\frac{6(x^2+y^2)}{\sqrt{x^2+y^2}}$$

in polari:
$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0,3] \\ y = \rho\sin(\theta) & \theta \in [0, 2\pi] \end{cases}$$

$$\implies \int_0^3 \int_0^{2\pi} -6\rho^2 \, d\theta \, d\rho = -12\pi \int_0^3 \rho^2 \, d\rho = -12\pi \left.\frac{\rho^3}{3}\right|_0^3 =$$

$$= -108\pi$$

$\blacksquare$

Esercizio:

Calcolare il flusso di $F(x,y,z) = \left(\frac{1}{4}x, \frac{9}{2}y, z+9\right)$ attraverso la superficie $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \mid \dots \right\}$

[illegible]

superficie $\Sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; z = 4x^2 - \frac{1}{4}y^2 - 9 \,,\; x^2+y^2 \leq 4 \,,\; y \geq 0 \right\}$

orientata in modo che il versore normale a Σ formi un angolo acuto con il versore fondamentale dell'asse z

<u>SOL</u>

Parametrizzo Σ:

$$\psi(x,y) = \begin{cases} x = x \\ y = y \\ z = 4x^2 - \frac{1}{4}y^2 - 9 \end{cases}$$

$\psi_x = (1, 0, 8x)$

$\psi_y = (0, 1, -\frac{1}{2}y)$

$\psi_x \wedge \psi_y = (-8x, +\frac{1}{2}y, 1)$

Notiamo che $\langle N, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle = \begin{pmatrix} -8x \\ +\frac{1}{2}y \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1 > 0 \Rightarrow$ è acuto

Si ha:

$$\int_{\partial\Sigma} F \cdot n = \int_{\Sigma} F(\psi) \cdot n = \int_{\Sigma} \begin{pmatrix} \frac{1}{4}x \\ \frac{9}{2}y \\ 4x^2 - \frac{1}{4}y^2 \end{pmatrix} \begin{pmatrix} -8x \\ \frac{1}{2}y \\ 1 \end{pmatrix} =$$

$$= \int_{\Omega} -2x^2 + \frac{9}{4}y^2 + 4x^2 - \frac{1}{4}y^2 \, dx\,dy =$$

$$= \int_{\Omega} 2x^2 + 2y^2 \, dx\,dy = 2\int_{\Omega} x^2 + y^2 \, dx\,dy$$

in polari: $\begin{cases} x = \rho\cos(\theta) & \rho \in [0,2] \end{cases}$

$\{ y = \rho \sin(\theta) \quad \theta \in [0, \pi]$

$$\Rightarrow \; 2 \int_0^2 \int_0^\pi \rho^3 \, d\theta \, d\rho = 2\pi \left(\frac{\rho^4}{4} \right) \Big|_0^2 = 8\pi$$

<u>Esercizio</u>:

Calcolare il flusso entrante del campo vettoriale

$$F(x,y,z) = (4e^x, 2y^3 z, -3y^2 z^2)$$

dal bordo di $\Omega = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, 0 \le x \le 4 - y^2 - z^2, \; y \ge 0 \right\}$

<u>SOL</u>

<u>1° Metodo</u>:

Suddividendo Ω in tre insiemi $\Sigma_1, \Sigma_2, \Sigma_3$:

$$\Sigma_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x = 0, \; y^2 + z^2 \le 4, \; y \ge 0 \right\}$$

$$\Sigma_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x = 4 - y^2 - z^2, \; y^2 + z^2 \le 4, \; y \ge 0 \right\}$$

$$\Sigma_3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, y = 0, \; 0 \le x \le 4 - z^2, \; -2 \le z \le 2 \right\}$$

Parametrizzo $\Sigma_1, \Sigma_2, \Sigma_3$:

$$\varphi(y,z) = \begin{cases} x = 0 \\ y = y \\ z = z \end{cases}, \quad \psi(y,z) = \begin{cases} x = 4 - y^2 - z^2 \\ y = y \\ z = z \end{cases}, \quad \Gamma(x,z) = \begin{cases} x = x \\ y = 0 \\ z = z \end{cases}$$

$$\varphi_y = (0, 1, 0) \qquad \psi_y = (-2y, 1, 0) \qquad \Gamma_x = (1, 0, 0)$$

$\varphi_y = (0, 1, 0)$ $\qquad \psi_y = (-2y, 1, 0)$ $\qquad \Gamma_x = (1, 0, 0)$

$\varphi_z = (0, 0, 1)$ $\qquad \psi_z = (-2z, 0, 1)$ $\qquad \Gamma_z = (0, 0, 1)$

$\varphi_y \wedge \varphi_z = (1, 0, 0)$ $\qquad \psi_y \wedge \psi_z = (1, 2y, 2z)$ $\qquad \Gamma_x \wedge \Gamma_z = (0, -1, 0)$

Si ha:

$$\int_{\Sigma_1} F \cdot n = \int_{\Sigma_1} F(\varphi) \cdot N_1 = \int_D \begin{pmatrix} 4 \\ 2y^3 z \\ -3y^2 z^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \int_D 4 \, dy \, dz =$$

$$= 4 \int_\Omega dy \, dz$$

in polari:
$$\begin{cases} y = \rho \cos(\theta) & \rho \in [0, 2] \\ z = \rho \sin(\theta) & \theta \in [\frac{3}{2}\pi, \frac{5}{2}\pi] \end{cases}$$

$$\Rightarrow 4 \int_0^2 \int_{\frac{3}{2}\pi}^{\frac{5}{2}\pi} \rho \, d\theta \, d\rho = 4\pi \int_0^2 \rho = 4\pi \left(\frac{\rho^2}{2} \right) \Big|_0^2 = 8\pi$$

$$\int_{\Sigma_2} F \cdot n = \int_{\Sigma_2} F(\psi) \cdot (-N_2) = \int_{\Sigma_2} \begin{pmatrix} 4 e^{4-y^2-z^2} \\ 2y^3 z \\ -3y^2 z^2 \end{pmatrix} \begin{pmatrix} -1 \\ -2y \\ -2z \end{pmatrix} =$$

$$= \int_D -4 e^{4-y^2-z^2} - 4y^4 z + 6y^2 z^3 \, dy \, dz$$

in polari:
$$\begin{cases} y = \rho \cos(\theta) & \rho \in [0, 2] \\ z = \rho \sin(\theta) & \theta \in [\frac{3}{2}\pi, \frac{5}{2}\pi] \end{cases}$$

$$\Rightarrow \int^2 \int^{\frac{5}{2}\pi} -4\rho e^{4-\rho^2} - 4\rho^6 \cos^4(\theta) \sin(\theta) + 6\rho^6 \cos^2(\theta) \sin^3(\theta) \, d\theta \, d\rho$$

$$\Rightarrow \int_0^2 \int_{\frac{3}{2}\pi} -4\rho e^{4-\rho^2} - 4\rho^6 \cos^4(\theta)\sin(\theta) + 6\rho^6 \cos^2(\theta)\sin^3(\theta) \; d\theta \, d\rho$$

$$= \int_0^2 -4\rho e^{4-\rho^2}\cdot\theta + \cancel{4\rho^6 \frac{\cos^5(\theta)}{5}} + 6\rho^6\left(\cancel{\frac{1}{4}\cos(\theta)\sin^4(\theta)} \cancel{-\frac{5}{32}\cos(\theta)} \cancel{+\frac{5}{192}\cos(3\theta)} \cancel{-\frac{1}{320}\cos(5\theta)}\right)\Bigg|_{\frac{3}{2}\pi}^{\frac{5}{2}\pi} =$$

$$= \int_0^2 -4\rho e^{4-\rho^2}\pi \, d\rho = -4\pi \int_0^2 \rho e^{4-\rho^2} d\rho = -4\pi\left(-\frac{1}{2}e^{4-\rho^2}\right)\Bigg|_0^2 =$$

$$= -4\pi\left(-\frac{1}{2}+\frac{1}{2}e^4\right) = 2\pi - 2\pi e^4 = 2\pi(1-e^4)$$

$$\int_{\Sigma_3} F\cdot n = \int_{\Sigma_3} F(\Gamma)\cdot(-N_3) = \int_{\Sigma_3}\begin{pmatrix}4e^x\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \int_D 0 \; dx\,dz = 0$$

In conclusione

$$\int_{\partial\Omega} F\cdot n = 8\pi + 2\pi(1-e^4)$$

<u>2° metodo</u>

$$\int_{\partial\Omega} F\cdot n = \int_\Omega \operatorname{div}(F)\,dx\,dy\,dz = \int_\Omega 4e^x + 6y^2z - 6yz^2 \; dx\,dy\,dz =$$

$0 \leq x \leq 4-y^2-z^2$

$0 \leq 4-y^2-z^2 \Rightarrow 0 \leq y \leq \sqrt{4-z^2}$

$0 \leq \sqrt{4-z^2} \Rightarrow z^2 \leq 4 \Rightarrow -2 \leq z \leq 2$

$$\Rightarrow \int_{-2}^{2}\int_{0}^{\sqrt{4-z^2}}\int_{0}^{4-y^2-z^2} 4e^x+6y^2z-6yz^2 \, dx\, dy\, dz =$$

$$= \int_{-2}^{2}\int_{0}^{\sqrt{4-z^2}} 4e^x+6xy^2z-6xyz^2 \Big|_0^{4-y^2-z^2} dy\, dz =$$

$$= \int_{-2}^{2}\int_{0}^{\sqrt{4-z^2}} 4e^{4-y^2-z^2}-4+6y^2z(4-y^2-z^2)-6yz^2(4-y^2-z^2)\, dy\, dz =$$

$$= \int_{-2} 4e^{4-(y^2+z^2)}-4+6y^2z(4-(y^2+z^2))-6yz^2(4-(y^2+z^2))\, dy\, dz$$

Passiamo in polari: $\begin{cases} y = \rho\cos(\theta) & \rho\in[0,2] \\ z = \rho\sin(\theta) & \theta\in[\frac{3}{2}\pi, \frac{5}{2}\pi] \end{cases}$

$\Rightarrow \ldots$ *come prima* $\Rightarrow \ldots = 10\pi - 2\pi e^4$

Esercizio:

Calcolare il flusso uscente del campo vettoriale

$$F(x,y,z) = \left(-\frac{3z}{\sqrt{x^2+y^2+z^2}}, -\frac{3}{\sqrt{x^2+y^2+z^2}}, \frac{3x}{\sqrt{x^2+y^2+z^2}}\right)$$

dal bordo di $\Omega = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \,\middle|\, x^2+y^2+z^2\le 76,\ y\ge 2\right\}$

SOL

$\Rightarrow \quad 2\le y\le\sqrt{76-x^2-z^2}\ ,\ \sqrt{76-x^2-z^2}\ge 2 \Rightarrow 76-x^2-z^2\ge 4 \Rightarrow x^2+z^2\le 72$

Posso suddividere Ω in due insiemi Σ_1, Σ_2:

$$\Sigma_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, y = 2,\ x^2 + z^2 \leq 12 \right\}$$

$$\Sigma_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, y = \sqrt{16 - x^2 - z^2},\ x^2 + z^2 \leq 12 \right\}$$

Parametrizzo Σ_1 e Σ_2:

$$\psi(x,z) = \begin{cases} x = x \\ y = 2 \\ z = z \end{cases} \quad , \quad \varphi(x,z) = \begin{cases} x = x \\ y = \sqrt{16 - x^2 - z^2} \\ z = z \end{cases}$$

$$\psi_x = (1, 0, 0) \qquad \varphi_x = \left(1, -\frac{x}{\sqrt{16 - x^2 - z^2}}, 0\right)$$

$$\psi_z = (0, 0, 1) \qquad \varphi_z = \left(0, -\frac{z}{\sqrt{16 - x^2 - z^2}}, 1\right)$$

$$\psi_x \wedge \psi_z = (0, -1, 0) \qquad \varphi_x \wedge \varphi_z = \left(-\frac{x}{\sqrt{16 - x^2 - z^2}}, -1, -\frac{z}{\sqrt{16 - x^2 - z^2}}\right)$$

Si ha:

$$\int_{\Sigma_1} F \cdot n = \int_{\Sigma_1} F(\psi) \cdot N_1 = \int_{\Sigma_1} \left(-\frac{3z}{\sqrt{x^2+z^2+4}}, -\frac{3}{\sqrt{x^2+z^2+4}}, \frac{3x}{\sqrt{x^2+z^2+4}}\right) \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} =$$

$$= \int_D \frac{3}{\sqrt{x^2+z^2+4}} = 3 \int_D \frac{1}{\sqrt{4+x^2+z^2}} \, dx \, dz =$$

in polari: $\begin{cases} x = \rho\cos(\theta) & \rho \in [0, \sqrt{12}] \\ z = \rho\sin(\theta) & \theta \in [0, 2\pi] \end{cases}$

$$= 3 \int_0^{\sqrt{12}} \int_0^{2\pi} \frac{\rho}{\sqrt{4+\rho^2}} \, d\theta \, d\rho = 6\pi \int_0^{\sqrt{12}} \frac{\rho}{\sqrt{4+\rho^2}} \, d\rho =$$

$$= 6\pi \left(\sqrt{4+\rho^2}\right) \Big|_0^{\sqrt{12}} = 6\pi (4 - 2) = 12\pi$$

$$= 6\pi \left(\sqrt{4+\rho^2}\right)\Big|_0^{\dots} = 6\pi\,(4-2) = 12\pi$$

Ora

$$\int_{\Sigma_2} F\cdot n = \int_{\Sigma_2} F(\psi)\cdot N_2 = \int_{\Sigma_2} \begin{pmatrix} -\frac{3z}{4} \\ -\frac{3}{4} \\ \frac{3}{4}x \end{pmatrix} \left(-\frac{x}{\sqrt{16-x^2-z^2}},\ -1,\ -\frac{z}{\sqrt{16-x^2-z^2}}\right) =$$

$$= \int_D \cancel{\frac{3xz}{4\sqrt{16-x^2-z^2}}} + \frac{3}{4} - \cancel{\frac{3xz}{4\sqrt{16-x^2-z^2}}}\ dx\,dz =$$

$$= \int_D \frac{3}{4}\,dx\,dz$$

in polari:

$$\begin{cases} x = \rho\cos(\theta) & \rho \in [0, \sqrt{12}] \\ z = \rho\sin(\theta) & \theta \in [0, 2\pi) \end{cases}$$

$$\Rightarrow \frac{3}{4}\int_0^{\sqrt{12}}\int_0^{2\pi} \rho\, d\theta\, d\rho = \frac{3}{2}\pi \left(\frac{\rho^2}{2}\right)\Big|_0^{\sqrt{12}} = 9\pi$$

In conclusione

$$\int_{\partial\Omega} F\cdot n = 12\pi + 9\pi = 21\pi$$

Esercizio:

Calcolare il flusso del campo vettoriale

$$F(x,y,z) = \left(\frac{2x}{x^2+y^2},\ \frac{3y}{x^2+y^2},\ 1\right)$$